

diagonals of real symmetric matrices of given spectra as a measure space

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Abstract

The set of diagonals of real symmetric matrices of given non negative spectrum is endowed with a measure which is obtained by the push forward of the Haar measure of the real orthogonal group.

We prove that the Radon Nicodym derivation of this measure with respect to the relative Euclidean measure is approximated by the coefficients of a sequence of zonal sphere polynomials corresponding with the given spectrum. There is a striking similarity between the role of the zonal sphere polynomials in the orthogonal case, and that of the Schur function in the Hermitian case.

Following this we obtain a combinatorial approximation for the probability of real symmetric matrix of a given spectrum to appear as the sum of two real symmetric matrices, each of a given spectrum. In addition we obtain a real orthogonal analogue to the Zuber Itzykson Harish Chandra integration formula.

Preliminaries and main result

We use Greek letters to denote partitions of unity,

$\delta = \delta_1, \delta_2, \dots, \delta_n$ where $\delta_i \geq 0$ and $\sum_i \delta_i = 1$, as well as for partitions of integers, $\gamma \vdash N$.

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a positive real vector so that $\sum \lambda_i = 1$

For simplicity, we assume that all λ_i are rational.

Define a diagonal matrix by

$$D_\lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \tag{1}$$

Let O_n be the real orthogonal group of n by n matrices.

By the Horn-Schur theorem, the set

$$((oD_\lambda o^t)_{11}, (oD_\lambda o^t)_{22}, \dots, (oD_\lambda o^t)_{nn})/o \in O_n \quad (2)$$

is the permutahedron PH_λ , defined by the vector λ . That is the convex hull of all the vectors $(\lambda_{\sigma(1)}, \lambda_{\sigma(2)} \dots \lambda_{\sigma(n)})$ for $\sigma \in S_n$.

PH_λ is endowed with a measure, given by pushing forward of the Haar measure of the real orthogonal group. We denote this measure by DH_λ^O , (In the Hermitian case DH_λ^U (see [FG]))

We assume that the Radon Nicodým derivation of DH_λ^O with respect to the relative Euclidian measure of PH_λ exists, and is almost everywhere continuous. Furthermore the suspected points of the Radon Nicodým derivation function are included in several affine spaces, their intersections with PH_λ , is of positive codimension. The proof will be given elsewhere.

Remark As not as in our case, the Radon Nicodým derivation of DH_λ^U with respect to the relative Euclidian measure of PH_λ can be computed directly using representation theory, thereby avoiding any need for differential geometry. See [F G] for more details.

Let λ be a partition of a positive integer N ($\lambda \vdash N$). Further, let

$$Z_\lambda(X) = \sum_{\eta \vdash N} a_\eta X^\eta \quad (3)$$

denote the zonal sphere polynomial corresponding to the partition λ (see [Mac] section 7 and [J 1])

The next theorem is our main result.

Theorem 1. *Let $\lambda = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be a non negative rational vector so that $\sum \lambda_i = 1$, and let N be a positive integer, such that $N\lambda$ is a vector of*

integers.

Let $Z_{N\lambda}(X) = \sum_{\eta \vdash N} a_\eta^N X^\eta$ be the corresponding zonal sphere polynomial. Assume $\frac{\eta}{N}$ is a continuous point of the density function (Radon Nicodým derivation of DH_λ^o with respect to the relativ Euclidian measure on PH_λ). Then as N tends to infinity, $\frac{a_\eta^N}{Z_{N\lambda}(I_n)}$ (I_n is the corresponding unit matrixe) approximates this density function at the point $\frac{\eta}{N}$ in PH_λ .

Remark Compare to the characterization of the zonal spher polynomials coefficient in [Mac 2.27]page 409

Remark In the Hermitian case the zonal sphere polynomials are replaced by the Schur functions in order to obtain a similar approximation. See[F G] for more details

Corollaries of the main result

Theorem 1 leads to two corollaries we introduce now

Let $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ be another vector of rational non negative numbers such that $\sum \gamma_i = 1$, and $D_\gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$.

Let N be a large integer so that $N\lambda$ and $N\gamma$ are integral vectors . Let

$$Z_{N\lambda}(X)Z_{N\gamma}(X) = \sum_{\eta \vdash 2N} C_\eta^{N\lambda, N\gamma} Z_\eta(X) \quad (4)$$

be the linearization of the product of the two zonal sphere polynomials.(see [Mac] page 409)

Claim 1. *Using the Haar measure of the orthogonal group as a probability measure, the Radon-Nikodym derivative at $\frac{\eta}{N}$ of the distribution of the random variable, $\frac{\eta'}{N}$ satisfying the equation $\frac{oD_\lambda o^t + o'D_\gamma o'^t}{2} = D_{\frac{\eta'}{N}} : o, o' \in O_n$, with respect to the relative Euclidian measure on the simplex, is approximated, up to normalization, by $C_\eta^{N\lambda, N\gamma}$.*

Simply speaking, the probability to obtain a matrix with spectra close to $\frac{\eta}{N}; \eta \vdash N$, as the sum of two symmetric matrices of spectra λ and γ , chosen randomly with respect to the Haar measure of the real orthogonal group, can be evaluated by the linearization coefficients as above

The reason for this is that one can define a measure on the set $\{\frac{1}{2}((D_\lambda^o + D_\gamma^{o'})_{11}, (D_\lambda^o + D_\gamma^{o'})_{22}, \dots, (D_\lambda^o + D_\gamma^{o'})_{nn})\}$ (using the notation $A^o \equiv oAo^t$), given by the push forward the Haar measure of $O_n \times O_n$. Since this measure is $DH_\lambda^o * DH_\gamma^o$ (the convolution of the two measures), the main result tells us that it can be approximated by the coefficients of the product $Z_{N\lambda}(X)Z_{N\gamma}(X)$

Remark In the Hermitian case, the Littlewood Richardson coefficients replace the coefficients $C_\eta^{N\lambda, N\gamma}$ in (4). See [F G] for more details

The second corollary deals with the real symmetric version of the Zuber Itzykson Harish Chandra integral.

Assume that C is a real symmetric matrix. For λ and D_λ as in (1), define

$$I_\lambda(C) \equiv \int \exp(\text{trace}(CD_\lambda^o)) dO \quad (5)$$

integrated over O_n with respect to its Haar measure.

Without loss of generality, one can assume that C is a diagonal matrix. Now, the integral may be taken over PH_λ , with respect to DH_λ^o .

$C_1, C_2 \dots C_n$ be the spectra of C , then for arbitrary N we have

$$I_\lambda(C) = \int_{PH_\lambda} \prod (\exp \frac{C_i}{N})^{N(D_\lambda^o)_{ii}} d(DH_\lambda^o) \quad (6)$$

By using Riemann-sums to approximate the integration over PH_λ , computing the integrand at the points of type $\frac{\gamma}{N} : \gamma \vdash N$ in PH_λ , and then by the main result approximate the density function of DH_λ^o at those points using the coefficients of the zonal sphere polynomial $Z_{N\lambda}(X)$ we obtain the following.

Claim 2. *If λ is a unit partition as given above, then*

$$I_\lambda(C) \sim \frac{1}{Z_{N\lambda}(I_n)} Z_{N\lambda}(\exp \frac{C_1}{N}, \exp \frac{C_2}{N}, \dots \exp \frac{C_n}{N}) \quad (7)$$

Remark One can ignore the points of PH_λ in which the Radon Nicodym derivation is not continues thanks to our assumptions on DH_λ .

Remark In the Hermitian version of this process, one may replace the zonal sphere polynomial in the right hand side with the Schur function with respect to the same parametrization. The very elegant determinant formula of Harish Chandra Zuber Itzykson is accepted as the limit of the determinants formula for the corresponding sequence of Schur functions. Unfortunately, the zonal sphere polynomial, do not yield such nice expression for their computations

Proof of the main result

Proof. Let us define a sequence of polynomials as follows:

$$P_\lambda^N(X) = \int_{O_n} \text{trace}^N(D_\lambda^o D_X) do \quad (8)$$

once integrated over the orthogonal group with respect to its Haar measure, and X is a vector of indeterminants $X_1, X_2, \dots X_n$.

Remark In [J 2] Identity

$$P_\lambda^N(X) = \sum_{\eta \vdash N} c_\eta Z_\eta(D_X) Z_\eta(D_\lambda) \quad (9)$$

was obtained, and the coefficients c_η were investigated .However we avoid it in our treatment .

Now we rewrite (8)as

$$\text{trace}^N(D_\lambda^o D_X) = (\sum (D_\lambda^o)_{ii} X_i)^N \quad (10)$$

One can observe the similarity between the sequence of polynomials, $P_\lambda^N(X)$, and the sequence of Bernstein polynomials approximating the density of

DH_λ^o (see [Lo] page 54)

More explicitly, by writing

$$P_\lambda^N(X) = \sum_{\eta \vdash N} a_\eta^N X^\eta \quad (11)$$

and combining with (10), we obtain

$$a_\eta^N = \int \binom{N}{\eta} \prod_i (D_\lambda^o)_{ii}^{\eta_i} dO \quad (12)$$

Where the integral is over the orthogonal group with respect to its Haar measure.

Let $Y = Y_1, Y_2, \dots, Y_n$ be a real vector such that $\sum_i Y_i = 1 : Y_i \geq 0$.

The next polynomials

$$\binom{N}{\eta} \prod_i Y_i^{\eta_i} \quad (13)$$

are special members of the family of "Dirichlet distributions", so the integrand of equation (12) tends to $\delta_{\frac{\eta}{N}}$ (the delta function concentrated around the points $\frac{\eta}{N}$) over the $n-1$ dimensional simplex, as N tends to infinity

Therefore, as N tends to infinity, one finds that the coefficients a_η^N tend to the density of DH_λ^o at the points $\frac{\eta}{N}$ in which it is continuous.

Next, we treat the left hand side of (10) by writing

$$trace^N(D_\lambda^o D_X) = trace((D_\lambda^o D_X)^{\otimes N}) \quad (14)$$

Let V denotes an n dimensional vector space over the reals, and let v_1, v_2, \dots, v_n be an orthogonal basis of V .

Let us define the η weight space for any partition η of N , of no more than n parts, by the formula

$$V^{\otimes N}_\eta = span\{v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_N} : \#\{l : i_l = j\} = \eta_j\} \quad (15)$$

Now because D_λ is a diagonal matrix, we have

$$trace^N(D_\lambda D_X) = trace((D_\lambda D_X)^{\otimes N}) = \sum_{\eta} trace(D_\lambda D_X)^{\otimes N}|_{V^{\otimes N}_\eta} \quad (16)$$

and

$$trace((D_\lambda D_X)^{\otimes N})|_{V^{\otimes N}_\eta} = \binom{N}{\eta} \prod \lambda_i^{\eta_i} X_i^{\eta_i} \quad (17)$$

Now, using spectral factorization of real symmetric operators, one obtain that

$$(D_\lambda)^{\otimes N} = \sum_{\eta \vdash N} \gamma_\eta \sum \alpha_\gamma^t \alpha_\gamma \quad (18)$$

where the second summation runs over a basis of the the eigenspace of weight η with eigenvalue $\gamma_\eta = \prod \lambda_i^{\eta_i}$,
(One can choose the basis as the standard unit vectors in $V_\eta^{\otimes N}$).

Now,one can use the central limit law (using equation (17)) to write

$$P_\lambda^N(X) \sim \int_{O_n} \sum_{\eta \vdash N} \gamma_\eta \sum_{\gamma} trace((o^{\otimes N} \alpha_\gamma)(o^{\otimes N} \alpha_\gamma)^t D_X^{\otimes N}) do \quad (19)$$

summed over $\eta \vdash N$ such that $\frac{\eta}{N} \sim \lambda$.

The first \sim in the equation means that contribution of the other summands to the coefficients a_η^N vanishes to zero. The second \sim denotes the closeness of real vectors at any coordinate.

To get closer to our goal one may apply representation theory of $Gl_n(R)$

By the Schur Weyl duality we have

$$V^{\otimes N} = \bigoplus V_\eta \otimes M_\eta \quad (20)$$

where the sum is over $\eta \vdash N$ of no more than n parts,

and $V_\eta(M_\eta)$ are the $Gl_n(R)(S_N)$ simple modules in their actions on the

tensor product space. See [G W] page 138 for more details

The striking fact is that, as N tends to infinity

$$trace^N(D_\lambda D_X) \sim \sum_{\frac{\eta}{N} \sim \lambda} trace((D_\lambda D_X)^{\otimes N}|_{V_\eta \otimes M_\eta}) \quad (21)$$

(see the remarks on \sim after (19)).

The discovery of this concentration phenomena is, some times, attributed to Kyel and Werner and has been rediscovered several times. (See [Ch] for a nice proof of it).

Remark At this point, one can easily see the full Hermitian perspective of the problem. See [F G] for more details . For symmetric real matrices one has more work to do.

Using (19) and (21), one can write

$$P_\lambda^N(X) \sim \int \sum_{\gamma} c_\gamma (trace((o^{\otimes N} \alpha_\gamma)(o^{\otimes N} \alpha_\gamma)^t D_X^{\otimes N})) do \quad (22)$$

summed over a basis of eigenvectors (of eigenvalues c_γ) of $D_\lambda^{\otimes N}$ in $(V_{\delta'} \otimes M_{\delta'}) \cap (V^{\otimes N})_\delta$, where $\frac{\delta}{N}, \frac{\delta'}{N} \sim \lambda$.

Now, for each vector α in $V^{\otimes N}$, we define a polynomial by the formula

$$Q_\alpha(m) = trace(\alpha^t \alpha m^{\otimes N}) \quad (23)$$

for any n by n matrix m

Before introducing the next lemma, we quote the definition of zonal sphere polynomials from the first page of [J 2]

"... Denote by V_f the space of polynomials of degree f over the real symmetric positive definite n by n matrices.

$GL_n(R)$ acts on V_f as follows

(For $g \in GL_n(R), P \in V_f : g(P(m)) = P(gmg^t)$).

As $GL_n(R)$ module $V_f = \bigoplus_{(f)} V_{(f)}$ where $(f) = f_1, f_2 \dots f_n$ is a partition of

f and $V_{(f)}$ is the subspace of V_f on which $GL_n(R)$ acts irreducibly corresponding to the partition $2(f)$

The zonal polynomial Z_f spans the real orthogonal group invariant vector space in $V_f \dots$ ”

To complete the proof of Theorem 1 we need the following lemma

Lemma 1. *Using the notations from [J2], for α_γ as in (22), we have that $Q_{\alpha_\gamma}(m) \in \bigoplus V_{(f)}$ summed over (f) such that $\frac{(f)}{N} \sim \lambda$.*

Proof. One need to apply some basic representation theory of $gl_n(R)$, the Lie algebra of $GL_n(R)$. [G W] page 228 is good reference to refer. We note that the weights referred to in (15) and the weights in the context of Lie algebras representation theory coincide for an appropriate choice of basic positive roots vectors.

Now, since the map $\alpha \mapsto \alpha^t \alpha$ is bilinear, it can be factorized through the tensor product. $\alpha \mapsto \alpha \otimes \alpha \mapsto \alpha^t \alpha$.

Furthermore for $\delta \vdash N$ we have $V_\delta \otimes V_\delta = \bigoplus V_\omega$, summed over $\omega \preceq 2\delta$, where \preceq is the dominant order defined on the weight lattice (see page 237 in [G W])

Since α_γ is of weight δ' , $\alpha_\gamma \otimes \alpha_\gamma$ is of weight $2\delta'$, and so belongs to $\bigoplus V_\omega$ summed on ω in the next small fragment: $2N(\lambda - \epsilon) \preceq 2\delta' \preceq \omega \preceq 2N(\lambda + \epsilon')$, where ϵ, ϵ' are arbitrarily small positive vector, as N tends to infinity. The central inequality just expressed the fact that the weights of vectors in a highest weight modules are smaller than its highest weight. \square

Given Lemma 1, Theorem 1 is now proved since the integral in (22) is just a projection onto the space spanned by the unique zonal polynomial in each $V(f)$. \square

Numerical Experiments:

We test the 3×3 case for the diagonal matrix $\text{diag}(0, 1, 2)$, conjugated by (a) $SO(3)$, and (b) $SU(3)$ (Figure 1). In each case we sample 30,000 conjugations, and compute the resulting diagonal. In both cases we have a two-dimensional distribution, based on the $(1, 1)$ and $(2, 2)$ positions of the diagonal. The $(3, 3)$ entry is determined from the first two by the trace condition. .

We have also test conjugations of rank 1 matrices

We sample the diagonals of a 3×3 matrix of rank 1 conjugated by $SO(3)$ random matrices (See Fig 2 (a)) On the other hand we sample the zonal sphere polynomial's coefficients corresponding to one row (using example 1 of [Mac]Page 410)(See Fig 2 (b))

In Figure 3 the coefficients of the zonal sphere polynomial of two variables corresponding to one row (the red points), have been compared with the function; $xy^{-\frac{1}{2}}, x + y = \text{constant}$, which is computed in [FG] explicitly for two by two symmetric matrices for that case. In the figures, we use the notation J_s^r as employed in [Mac, p. 385] for the Jack-polynomials. Note that the $r = 2$ case corresponds to the zonal polynomials.

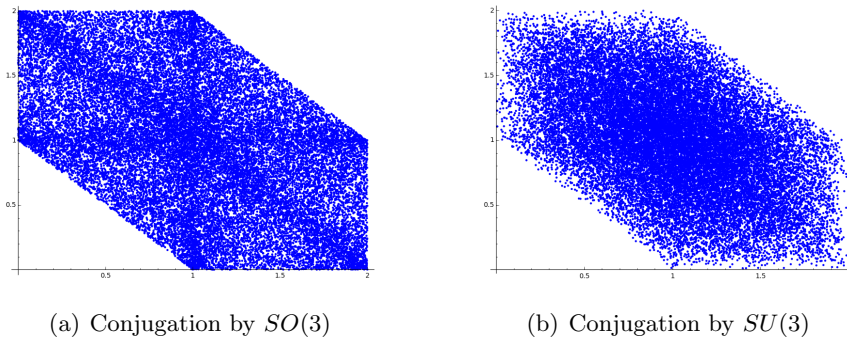


Figure 1:

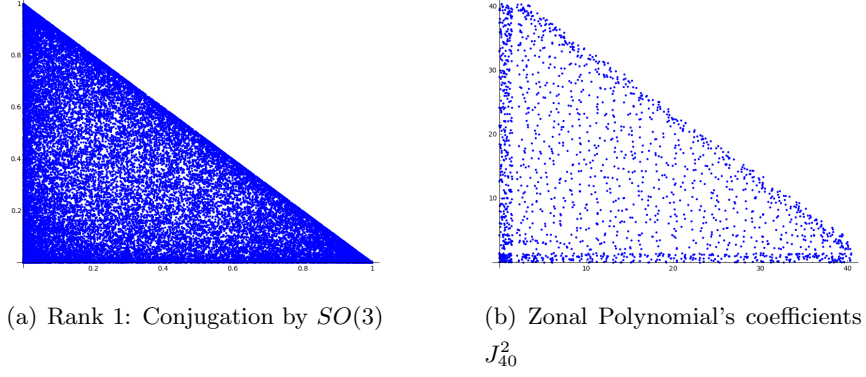


Figure 2:

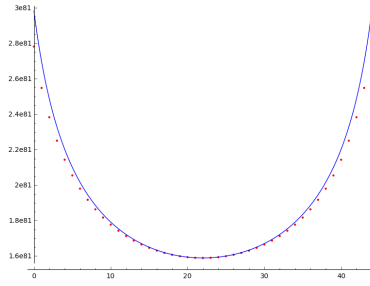


Figure 3: Zonal coefficients of J_{50}^2 (dotted) against theoretical prediction

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